International Journal of Theoretical Physics, Vol. 12, No. 1 (1975), pp. 61-67

On Nambu's Generalized Hamiltonian Mechanics

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Received: 27 *July* 1974

Abstract

Recently, Y. Nambu stated the principles of a new analytical mechanics which allows an odd, as well as an even, number of phase space variables. In this paper we investigate if this mechanics is independent for the odd case of Dirac's classical mechanics because there are reasons which allow one to suspect that independence fails. We prove that Nambu's mechanics is independent of Dirac's mechanics, at least for the odd cases, so that the Nambu mechanics can, in principle, describe physical systems for which the Dirac mechanics is not suitable.

As regards the even case, it is natural to confront the Nambu mechanics with the Hamiltonian one, because both have the same dimension for their phase spaces. In this paper we restrict our study to the comparison of the canonical groups of both formalisms and prove that both groups are different.

García Sucre and Kálnay have suggested that perhaps the Nambu mechanics is the natural one for quarks. We end our study with a brief discussion on this subject whose conclusion seems to support their conjecture.

1. *Introduction*

Nambu (1973) created a new analytical mechanics in which the number n of phase space variables $x_1, x_2, \ldots, x_n$ (usually even because to each q_i corresponds a p_i) can be any integer greater than one. This increases the possibility that, after quantisation, microsystems which are not so easy to tackle (e.g. some particles related to high energy physics) find in Nambu's mechanics the proper formalism for their description. In this direction García Sucre & Kálnay (1974) have recently shown that the quark statistics is consistent with Nambu's mechanics.

The main part of this paper is devoted to the c-number Nambu mechanics. The main purpose is to look for the possible relationship among Nambu and other mechanics in the cases where intuitively one could suspect that perhaps they are not independent. We shall first summarise the principles of this formalism (Nambu, 1973). Let

$$x = (x_1, x_2, \ldots, x_n), \qquad n = 2, 3, 4, \ldots \tag{1.1}$$

be the phase space variables of Nambu's mechanics. He assumes the existence for a given physical system on $n - 1$ constants of motion, which he called, Hamiltonians $H_1, H_2, \ldots, H_{n-1}$, and he defines a bracket

$$\{f_1, f_2, \ldots, f_n\} =^{df} \frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)} \tag{1.2}$$

It follows that the fundamental bracket relation is

$$\{x_1, x_2, \ldots, x_n\} = 1 \tag{1.3}$$

He postulates the equations of motion of a dynamical variable $f(x)$ to be

$$\dot{f} = \{f, H_1, H_2, \ldots, H_{n-1}\} \tag{1.4}$$

The Nambu formalism† can be confronted by the Hamiltonian one. We are expressing both formalisms in their bracket form. Let

$$q = (q_1, q_2, \ldots, q_N), \qquad p = (p_1, p_2, \ldots, p_N), \qquad N = 1, 2, 3, \ldots \tag{1.5}$$

the $2N$ phase space variables of the standard Hamiltonian formalism. As is well known, from the Poisson bracket

$$\{f, g\}_- = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \tag{1.6}$$

the fundamental bracket relations

$$\{q_i, q_j\}_- = \{p_i, p_j\}_- = 0 \tag{1.7a}$$

$$\{q_i, p_j\}_- = \delta_{ij} \tag{1.7b}$$

follow. The Hamilton equations of motion in Poisson bracket notation for a dynamical variable $f(q, p)$ are

$$\dot{f} = \{f, H\}_- \tag{1.8}$$

The relations (1.5) to (1.8) (with the exception of (1.7a)) correspond respectively to (1.1) to (1.4). The relations (1.7a) correspond to formulas like $\{x_1, x_2, x_3, x_4, \ldots, x_N\} = 0$ which follow directly from equation (1.2). *Moreover, for $n = 2$ it is transparent that the Nambu mechanics is totally equivalent through the identification $x_1 = q_1$, $x_2 = p_1$ to the Hamiltonian mechanics with $N = 1$*, so that, in this sense, *the Nambu mechanics can be considered as a*

† The Nambu formalism considered in the present paper is the corresponding one to his equation (4) (which is equivalent to our (1.4)) and not to the variants proposed by Nambu in his equation (6) or (7). All variants coincide for the *triplet case* ($n = 3$) which is the main case developed by Nambu in his article.

natural generalisation of Hamilton mechanics for a state vector space of arbitrary dimensionality.†

In this paper we shall discuss two of the questions which have resulted from Nambu's paper. (i) Is Nambu formalism for *odd* number n of phase space variables equivalent to a Hamiltonian formalism with $2N$ phase space variables restricted by suitable phase space constraints? In other words, *is Nambu mechanics for odd n equivalent to the classical Dirac mechanics?* (ii) *Is the canonical group of the Nambu formalism for even $n > 2$ equivalent to the canonical group of a standard Hamiltonian systems of $2N = n$ variables?* These questions will be treated respectively in Sections 2 and 3. The answers become negative in both cases. In the fourth and last section we shall consider the eventual possibility of the description of quarks by means of Nambu's mechanics, but still in a very preliminary way.

Remark. We stress that Nambu's equation of motion (1.4) is a first-order differential equation with regards to time. As a result, the state vector space of Nambu c-number theory is represented by his n-dimensional phase space of coordinates x. With the exception of constraints imposed on the x's which we shall not concern ourselves with in this paper. We shall refer to this remark several times.

2. *On Odd Dimensional Phase Space*

In this section we consider the c-number form of Nambu's mechanics when applied to systems whose phase space is odd dimensional; and, discuss if it is different to Dirac's classical mechanics, which can only be applied (at least in its present form) to the standard case of even number $2N$ of coordinates in phase space. The question of the relation between these mechanics arises because Dirac mechanics is a Hamiltonian mechanics for systems whose $2N$ coordinates (1.5) are restricted by phase space constraints $\phi_a(q, p)$

$$\phi_a \approx 0 \tag{2.1‡}$$

so that it could happen, at least in principle, that these restrictions could decrease to a lower *odd* number n', the dimension of the state vector space. If $n' = n$, the dimension of the state vector spaces in Dirac's and Nambu's theories would coincide; then the next question would be to compare the structure of both dynamical theories; if not, both mechanics could not be equivalent. We anticipate that we shall show that the dimension n' of the state vector space for Dirac theory cannot be odd, and this shows that *Nambu's mechanics, is inequivalent to Dirac's mechanics, at least in the case in which the Nambu mechanics is more surprising, i.e. in the case of odd n.*

† Martin (1959) also considered a c-number mechanics where the phase space coordinates could be odd in number. For a relation of this mechanics to the Nambu formalism, see Ruggeri (to be submitted to publication). On the other hand Nambu's article influenced one of us (I. Cohen, to be submitted for publication) to develop a generalisation of Nambu mechanics. In the frame of Cohen's work the relation between formalisms is also clarified.

‡ We use the Dirac (1964) notation for $\approx$.

We must now exhibit that n' must be even. But first we summarise the following from the Dirac theory (Dirac, 1950, 1958, 1964). He starts from the Lagrangian. The computation of the canonical momenta may imply a certain number of relations among the q's and the p's. From self-consistency new constraints may arise. We call C the total number of constraints. The set of all C constraints is further decomposed in the union of two disjoint sets: The set of the first- and the second-class constraints. We call respectively $C_{(1)}$ and $C_{(2)}$ the number of their elements. More generally, we have by definition that for any physical variable

$$g(q, p) =^{df} \text{first class variable iff } \{g, \phi_a\}_- = 0, \qquad a = 1, 2, \ldots, C \quad (2.2)$$

The first-class constraints are those first-class variables which are also constraints. The second-class constraints are those which belong to the complementary basic set. The number $C_{(2)}$ of second-class constraints must be even. There is no restriction on the number $C_{(1)}$ of first-class constraints.

We now begin our study of the comparison of the dimensions of the state vector spaces of a Nambu formalism of n independent phase space coordinates (1.1) compared to a Dirac formalism with $2N$ phase space coordinates (1.5) restricted by $C = C_{(1)} + C_{(2)}$ constraints (2.1). It is clear from Dirac's theory that each second-class constraint decreases by one the dimension of the state vector space in Dirac's formalism (dimension equal to $2N$ before introducing the first constraint). On the other hand, let us assume for the moment that each first-class constraint decreases by two the dimension of that space. After all constraints have been taken into account, it would be clear that the dimension n' of the state vector space in Dirac's theory is

$$n' = 2N - 2C_{(1)} - C_{(2)} \quad (2.3)$$

and that n' is even and therefore cannot be equal to n. The proof that the mechanics by Nambu and Dirac are different would be concluded.

Note. This result does not necessarily remain correct if the Nambu phase space is implemented with an additional number of auxiliary variables (Ruggeri, private communication).

We now look for the proof of the assumption that each first-class constraint decreases by two the dimension of Dirac's state vector space.

With the assumption for the mathematical conditions being equation (2.1) an implicit definition of p_a as a function of the remaining phase space coordinates,

$$p_a \approx \mu_a(q_1, \ldots, q_{a-1}, q_a, q_{a+1}, \ldots, q_N, p_1, \ldots, p_{a-1}, p_{a+1}, \ldots, p_N) \quad (2.4)$$

this point may be proven.

(If q_a (instead of p_a) can be implicitly defined, the proof would run in a similar way.) The proof is as follows: It is known (see equation (3.23) of Bergmann & Goldberg (1955), also Marx (1972) and pp. 46–47 of Cohen (1972)) that those dynamical variables which are true physical variables must have a null Poisson bracket with all first-class constraints. Let us call $\{g_r \mid r =$

$1, 2, \ldots\}$ a basis in the state vector space (which is the space of those variables). From equations (2.2) and (2.4) we have,

$$\frac{\partial g_r}{\partial q_a} \approx \sum_{b \neq a} \frac{\partial g_r}{\partial q_b} \frac{\partial \mu_a}{\partial p_b} - \sum_{a \neq b} \frac{\partial g_r}{\partial p_b} \frac{\partial \mu_a}{\partial q_b} \tag{2.5}$$

Let us consider in a $2N + 1$ space the hypersurface Σ_r formed by the points of coordinates $(q_1, \ldots, q_N, p_1, \ldots, p_N, g_r(q, p))$, which is a $2N$ dimensional manifold. Let us call γ_r the $2N - 1$ dimensional hypersurface obtained by the intersection of Σ_r with the hyperplane $q_a = 0$. Equation (2.5), being a first-order differential equation in q_a; implying, that the whole Σ_r is uniquely determinated by γ_r, so that equation (2.5) lowers by one the dimension of the state vector space. Finally, equation (2.4) lowers again by one that dimension. □

Note. The proof is even more simple in the particular case $\mu_a = 0$, because then $p_a \approx 0$ and $\partial g_r/\partial q_a \approx 0$ so that the values that g_r takes in the state vector space (subspace of the phase space) only depend on the q_i, p_i with $i \neq a$. (See the related discussion of Dirac (1958).)†

3. *On Even Dimensional Phase Space*

Though mainly developed for phase space triplets, Nambu's mechanics is also possible for even dimensional phase spaces (Nambu, 1973) and then the question of the comparison of the Nambu mechanics with the standard Hamiltonian mechanics arises. We shall restrict the present study to the comparison of the canonical groups of both formalisms.

We are now considering phase space unconstrained systems for both Nambu and Hamiltonian mechanics; so that for each formalism we have that its state vector space is the space spanned by the respective phase space coordinates (cf. Remark at the end of the Introduction). Thus, a necessary condition for the equivalence of the mechanics to be compared is $n = 2N$.

As the case $n = 2N = 2$ was considered in the Introduction, we shall henceforth restrict ourselves to even $n \geqslant 4$. Up to a relabelling we can always put

$$x_1 = q_1, \qquad x_2 = p_1, \ldots, x_{2i-1} = q_i, \qquad x_{2i} = p_i, \ldots, x_{n-1} = q_N,$$
$$x_n = p_N, \qquad i = 1, 2, \ldots, N = n/2 \tag{3.1}$$

† We owe to Professor Ruggeri (private communication) the following interesting remark: Let us consider an ordinary Lagrangian problem: Time derivatives of order not greater than one, only holonomic constraints if any. When going to a phase space formalism one can only obtain the standard Hamiltonian formalism or the Dirac formalism, depending on the absence or presence of phase space constraints. In a comparison with the Nambu formalism, the Hamiltonian one must be discarded because of the obvious difference in dimensions of the state vector spaces. From the above results, it is seen that the same happens with the Dirac formalism, therefore there exist no such Lagrangian formalism whose equation of motion be consistent with the equation of motion of the odd dimensional Nambu formalism. (See note after equation (2.3).)

Canonical transformations $(q, p) \to (q', p')$ can be introduced in two equivalent ways in Hamiltonian mechanics. In the first form they are the transformations that leave invariant the Hamiltonian formalism, this is the more usual way. In the second form (see e.g. Kilmister, 1964), they are the transformations that leave invariant the values of the fundamental Poisson brackets (1.7). When going to a different formalism we prefer the second definition for the following reasons: (i) There are formalisms where one knows nothing about an action principle. Thus, given a function like $H'(q', p')$ introduced to the transformed system in order to have transformed equations of motion of the same form as the original ones, one does not know if H' is a Hamiltonian. For example: in Hamiltonian mechanics the transformation $q' = p$, $p' = q$ is known to be non-canonical; however, by defining $H'(q', p') = -H(q, p)$ one knows that the transformed equations of motions are of the Hamiltonian form. However, if H is related to the action according to its role as a Hamiltonian, the same does not happen of H'. Therefore, H' is not a proper Hamiltonian and the transformation is not canonical. If, in this example, one could not resort to the action principle (as may be the case in other formalisms) one could not know if the transformation is canonical or not. Things are much more transparent when the second definition is used: not only in the quoted example (where the second definition immediately tells that the transformation is anti-canonical instead of canonical), but in more general formalisms where usually a bracket (at least in the ordinary sense) may be defined but the existence of an action principle is obscured. (ii) Frequently the final aim of a c-number formalism is to quantise it. The quantum analogues of the canonical transformations are the unitary transformations and they preserve the quantum fundamental brackets. It is not easy to look for a quantum definition of canonicity like the first classical one.

These remarks are consistent with Nambu's procedure: He defines his canonical transformations as those $x \to x'$ that leave invariant the value of the fundamental Nambu bracket (1.3). We shall subscribe to Nambu's definition.

Proposition. For even $n = 2N \geqslant 4$ the Nambu canonical group, when translated to Hamiltonian variables through (3.1), is different from the Hamilton canonical group. More explicitly, there are Nambu canonical transformations which are not Hamilton canonical transformations.

Proof. Because of equation (1.2) the transposition of two Nambu coordinates only changes the sign of the Nambu bracket. Then the transformation

$$x_1' = x_2, \qquad x_2' = x_1, \qquad x_3' = x_4, \qquad x_4' = x_3, \qquad x_i' = x_i \;\forall i > 4 \tag{3.2}$$

is Nambu canonical. However, in Hamiltonian variables equation (3.2) reads

$$\begin{gathered} q_1' = p_1, \qquad p_1' = q_1, \qquad q_2' = p_2, \qquad p_2' = q_2, \\ q_i' = q_i, \qquad p_i' = p_i \forall i > 4 \end{gathered} \tag{3.3}$$

which is well known not to be Hamilton canonical.

4. *Again a Short Remark on Quarks*

García Sucre & Kálnay (1974) found an indication toward the possibility that quarks could be properly described in terms of the quantum Nambu mechanics. They proved that the quark statistics is consistent with a quantification of Nambu mechanics.

Here we present another tentative indication in the same direction: Let us consider a Nambu triplet x_1, x_2, x_3. The state vector space of this classical system is three-dimensional (cf. Remark at the end of the Introduction). But, this is just the dimension of the quantum vector space spanned by the basic triplet introduced in quark theory. Similarly, the three triplets considered in the quark-colour model could be compared with three Nambu triplets: equation (6) of Nambu (1973) just describes an integer number of Nambu triplets. These considerations may perhaps add another slight indication in support of the idea that quarks could find in Nambu mechanics its suitable formalism. Of course, the above is nothing but a rough guess, because of the deep differences between classical and quantum state vectors.

Acknowledgement

The authors are indebted to Professor G. Ruggeri for several critical discussions.

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